

Prolongation Approach to Bäcklund Transformation of Zhiber-Mikhailov-Shabat Equation

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The prolongation structure of Zhiber-Mikhailov-Shabat (ZMS) equation is studied by using Wahlquist-Estabrook's method. The Lax-pair for ZMS equation and Riccati equations for pseudopotentials are formulated respectively from linear and nonlinear realizations of the prolongation structure. Based on nonlinear realization of the prolongation structure, an auto-Bäcklund transformation of ZMS equation is obtained.

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I. INTRODUCTION

The off-conformally integrable models in two-dimensional space-time have some common features. They have spectrum-dependent Lax pairs, an infinite number of conserved currents and the underlying nonlinear symmetries, and can be solved by means of inverse scattering method. The $a_2^{(2)}$ Toda model, *i.e.* Zhiber-Mikhailov-Shabat (ZMS) model, is of such a fascinating class of integrable two-dimensional field theories. It is the third and the last relativistic single scalar Toda model (the others are Liouville and sine-Gordon models) [1,2], and has significant applications in physical context. To our knowledge, the equation of motion of ZMS model (ZMS equation) governs the propagation of resonant ultra-short plane wave optical pulses in certain *degenerate* media [3]. The integrability of ZMS model has been confirmed for a long time. The soliton solutions of the ZMS equation by means of inverse scattering method were given in [4]. The *S*-matrix approach to the quantum version of the model was seriously investigated by Izergin and Korepin [5] in terms of the quantum inverse scattering method, and by Smirnov [6] and Efthimiou [7] in the framework of perturbative conformal field theory. Recently, we have studied the infinitesimal dressing transformations and Lie-Poisson structure hidden in ZMS model [8]. Nevertheless, there would seem not to have a good knowledge of the finite nonlinear symmetries (such as the dressing group symmetry and Bäcklund transformation) of the model. In order to fill in the gap, we study the prolongation structure of ZMS equation in the spirit of Wahlquist-Estabrook's(W-E's) prolongation approach [9,10] in the present paper. Owing to the so-called prolongation structure, we discover an auto-Bäcklund transformation of ZMS equation, which turns out to be a set of Riccati-type differential equations of the so-called pseudopotentials.

II. PROLONGATION STRUCTURE OF ZMS EQUATION

W-E's prolongation structure is a very useful medium for searching Bäcklund transformation of nonlinear differential equations. To obtain the prolongation structure of ZMS equation:

$$\partial_+ \partial_- \phi - 2(e^\phi - e^{-2\phi}) = 0, \quad (2.1)$$

we define the following 2-forms on a four-dimensional manifold with coordinates associated with Eq.(2.1),

$$\begin{cases} \alpha_1 = d\phi \wedge dx^+ - \pi_\phi dx^- \wedge dx^+, \\ \alpha_2 = d\pi_\phi \wedge dx^- + 2(e^\phi - e^{-2\phi}) dx^- \wedge dx^+. \end{cases} \quad (2.2)$$

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These 2-forms constitute a closed ideal and would become null on the solution manifold $(\phi(x^+, x^-), \pi_\phi(x^+, x^-), x^+, x^-)$. They are the Pfaff forms of ZMS equation. For the above Pfaff forms we will assume that prolongation forms can be given by some 1-forms Ω^a ($a = 1, 2, 3, \dots, N$),

$$\Omega^a = -dq^a + F^a(\phi, \pi_\phi, q)dx^- + G^a(\phi, \pi_\phi, q)dx^+, \quad (2.3)$$

where N is an outstanding integer, q^a are the so-called pseudopotentials, and F^a and G^a are functionals of fields π_ϕ , ϕ and q^a .

The concept of pseudopotential plays a crucial role in the discussions of Bäcklund transformations and Lax pairs in W-E's prolongation method. As a matter of fact, the expected Bäcklund transformation and the first-order differential equations satisfied by Lax pair of ZMS Eq.(2.1) will be formulated as the differential equations for suitably defined pseudopotentials q^a . The integrability of pseudopotentials q^a requires that the ideal generated by the form sets $\{\alpha^a\}$ and $\{\Omega^a\}$ is closed, *i.e.*,

$$d\Omega^a = \eta_b^a \wedge \Omega^b + f^{a,i} \alpha_i, \quad (2.4)$$

where η_b^a and $f^{a,i}$ are some 1-forms and 0-forms respectively. When (2.4) is explicitly written out by using (2.2) and (2.3), it splits up into a set of partial differential equations:

$$\begin{aligned} \partial_\phi F^a &= 0, & \partial_{\pi_\phi} G^a &= 0, \\ F^b \partial_b G^a - G^b \partial_b F^a + \pi_\phi \partial_\phi G^a - 2(e^\phi - e^{-2\phi}) \partial_{\pi_\phi} F^a &= 0, \end{aligned} \quad (2.5)$$

where the derivative $\frac{\partial}{\partial q^a}$ is abbreviated to ∂_a . Analyzing these equations we find,

$$F^a = X_0^a + X_1^a \pi_\phi, \quad G^a = 2Y_0^a e^\phi + 2Y_1^a e^{-2\phi}. \quad (2.6)$$

In the ansatz (2.6) X_i^a and Y_i^a ($i = 0, 1$) are assumed to be functions of q^a only.

For the convenience of the later discussions we now introduce some vector fields (Lie derivatives) X_i and Y_i in N -dimensional space of pseudopotentials (q -space),

$$X_i = X_i^a \partial_a, \quad Y_i = Y_i^a \partial_a. \quad (2.7)$$

It is then a direct consequence of (2.5)-(2.7) that,

$$[X_0, Y_0] = X_1, \quad [X_0, Y_1] = -X_1, \quad [X_1, Y_0] = -Y_0, \quad [X_1, Y_1] = 2Y_1. \quad (2.8)$$

Because of the absence of $[X_0, X_1]$ and $[Y_0, Y_1]$, the set of Lie brackets given by (2.8) does not form a closed linear algebra. It is obviously impossible that one may close the algebra by setting the unknown commutators to be linear combinations of the given generators such that the results are consistent with the Jacobi identities. This is a big difference between the prolongation structure of ZMS equation and those of sine-Gordon equation, Ernst equation and chiral model [10,11]. However, the algebra can be closed by assigning new generators to the unknown commutators and repeating the process of working through the Jacobi identities. After a tedious but straightforward computation, we see that the enlarged algebra becomes an infinite-dimensional algebra $a_2^{(2)}$ (without center), which coincides with a well-known fact that ZMS model is the Toda field theory over the twisted Kac-Moody algebra $a_2^{(2)}$ [1,2]. Let $\{H_i^{(m)}, E_{\pm\alpha}^{(m)}\}$ be the Cartan-Weyl basis of $a_2^{(2)}$, which obey the following commutation relations:

$$\begin{aligned} [H_i^{(m)}, E_{\pm\beta}^{(n)}] &= \pm\delta_{i1}\delta_{\beta1}E_{\pm 1}^{(m+n)} \mp 2\delta_{i1}\delta_{\beta2}E_{\pm 2}^{(m+n)} \mp \delta_{i1}\delta_{\beta3}E_{\pm 3}^{(m+n)} \\ &\quad \pm 3\delta_{i2}\delta_{\beta1}E_{\mp 3}^{(m+n)} \mp 3\delta_{i2}\delta_{\beta3}E_{\mp 1}^{(m+n)}, \\ [E_{\pm\alpha}^{(m)}, E_{\pm\beta}^{(n)}] &= \mp\delta_{\alpha 1}\delta_{\beta 2}E_{\pm 3}^{(m+n)} \pm \delta_{\alpha 1}\delta_{\beta 3}H_2^{(m+n)}, \\ [E_\alpha^{(m)}, E_{-\beta}^{(n)}] &= (\delta_{\alpha 1}\delta_{\beta 1} - \delta_{\alpha 2}\delta_{\beta 2} - \delta_{\alpha 3}\delta_{\beta 3})H_1^{(m+n)} - \delta_{\alpha 3}\delta_{\beta 2}E_1^{(m+n)} \\ &\quad - \delta_{\alpha 2}\delta_{\beta 3}E_{-1}^{(m+n)} + 2\delta_{\alpha 3}\delta_{\beta 1}E_2^{(m+n)} + 2\delta_{\alpha 1}\delta_{\beta 3}E_{-2}^{(m+n)}, \end{aligned} \quad (2.9)$$

where ($m, n = 0, \pm 1, \pm 2, \pm 3, \dots$; $i = 1, 2$; $\alpha, \beta = 1, 2, 3$). Then we have the following identifications,

$$X_0 = E_1^{(-1)} + E_2^{(0)}, \quad X_1 = H_1^{(0)}, \quad Y_0 = E_{-1}^{(1)}, \quad Y_1 = E_{-2}^{(0)}. \quad (2.10)$$

Now we study the linear realizations of the vector fields X_i and Y_i in an infinite-dimensional q -space which has coordinate variables $\{q_j^{(m)}; j = 1, 2, 3; m = 0, \pm 1, \pm 2, \pm 3, \dots\}$. Following Omote [10], we introduce some auxiliary vector fields $\{A_{ij}^{(m)}\}$:

$$A_{ij}^{(m)} = \sum_{n=-\infty}^{+\infty} q_i^{(m+n)} \frac{\partial}{\partial q_j^{(n)}}. \quad (2.11)$$

They can be shown to satisfy commutator relations

$$[A_{ij}^{(m)}, A_{kl}^{(n)}] = \delta_{jk} A_{il}^{(m+n)} - \delta_{il} A_{kj}^{(m+n)}. \quad (2.12)$$

This fact implies that the set of vector fields $\{A_{ij}^{(m)}\}$ provide an operator version of graded matrices $\{e_{ij}^{(m)} = e_{ij} \otimes \lambda^m\}$ (λ is a gradation parameter). Therefore, the Cartan-Weyl basis of $a_2^{(2)}$ under consideration has the following linear realization [8] in a triplicated infinite-dimensional q -space:

$$\begin{cases} H_1^{(m)} = A_{11}^{(m)} - A_{33}^{(m)}, \\ H_2^{(m)} = A_{11}^{(m)} - 2A_{22}^{(m)} + A_{33}^{(m)}, \\ E_2^{(m)} = A_{31}^{(m)}, \\ E_{-2}^{(m)} = A_{13}^{(m)}, \end{cases} \quad \begin{cases} E_1^{(m)} = A_{12}^{(m)} - A_{23}^{(m)}, \\ E_{-1}^{(m)} = A_{21}^{(m)} - A_{32}^{(m)}, \\ E_3^{(m)} = A_{32}^{(m)} + A_{21}^{(m)}, \\ E_{-3}^{(m)} = A_{12}^{(m)} + A_{23}^{(m)}. \end{cases}$$

Explicitly,

$$\begin{cases} X_0 = \sum_{n=-\infty}^{+\infty} \left[q_1^{(n-1)} \frac{\partial}{\partial q_2^{(n)}} - q_2^{(n-1)} \frac{\partial}{\partial q_3^{(n)}} + q_3^{(n)} \frac{\partial}{\partial q_1^{(n)}} \right], \\ X_1 = \sum_{n=-\infty}^{+\infty} \left[q_1^{(n)} \frac{\partial}{\partial q_1^{(n)}} - q_3^{(n)} \frac{\partial}{\partial q_3^{(n)}} \right], \\ Y_0 = \sum_{n=-\infty}^{+\infty} \left[q_2^{(n+1)} \frac{\partial}{\partial q_1^{(n)}} - q_3^{(n+1)} \frac{\partial}{\partial q_2^{(n)}} \right], \\ Y_1 = \sum_{n=-\infty}^{+\infty} \left[q_1^{(n)} \frac{\partial}{\partial q_3^{(n)}} \right], \end{cases} \quad (2.13)$$

i.e., the components of F^a and G^a are assigned to the following linear representations in a triplicated infinite-dimensional q -space :

$$\begin{aligned} F_1^{(n)} &= q_3^{(n)} + q_1^{(n)} \pi_\phi, & F_2^{(n)} &= q_1^{(n-1)}, & F_3^{(n)} &= -q_2^{(n-1)} - q_3^{(n)} \pi_\phi, \\ G_1^{(n)} &= 2q_2^{(n+1)} e^\phi, & G_2^{(n)} &= -2q_3^{(n+1)} e^\phi, & G_3^{(n)} &= 2q_1^{(n)} e^{-2\phi}. \end{aligned} \quad (2.14)$$

Relying on the 1-form (2.3), we see that the pseudopotentials introduced in (2.14) satisfy equations

$$\begin{aligned} \partial_- \begin{bmatrix} q_1^{(n)} \\ q_2^{(n)} \\ q_3^{(n)} \end{bmatrix} &= \begin{bmatrix} \partial_- \phi & 0 & 1 \\ 0 & 0 & 0 \\ 0 & -1 & -\partial_- \phi \end{bmatrix} \begin{bmatrix} q_1^{(n)} \\ q_2^{(n-1)} \\ q_3^{(n)} \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} q_1^{(n-1)} \\ q_2^{(n)} \\ q_3^{(n)} \end{bmatrix}, \\ \partial_+ \begin{bmatrix} q_1^{(n)} \\ q_2^{(n)} \\ q_3^{(n)} \end{bmatrix} &= \begin{bmatrix} 0 & 2e^\phi & 0 \\ 0 & 0 & -2e^\phi \\ 2e^{-2\phi} & 0 & 0 \end{bmatrix} \begin{bmatrix} q_1^{(n)} \\ q_2^{(n+1)} \\ q_3^{(n+1)} \end{bmatrix}, \end{aligned} \quad (2.15)$$

on the solution surface $(\phi(x^+, x^-), \pi_\phi(x^+, x^-), x^+, x^-)$ of ZMS Eq.(2.1). Let us define a parameter-dependent potential $\Psi(\lambda)$ by

$$\Psi(\lambda) \equiv \begin{bmatrix} \psi_1(\lambda) \\ \psi_2(\lambda) \\ \psi_3(\lambda) \end{bmatrix} = \sum_{n=-\infty}^{+\infty} \lambda^n \begin{bmatrix} q_1^{(n)} \\ q_2^{(n)} \\ q_3^{(n)} \end{bmatrix}. \quad (2.16)$$

Then we get from (2.15) the partial differential equations for $\Psi(\lambda)$: $\partial_{\pm}\Psi = A_{\pm}\Psi$, where

$$\begin{cases} A_+ = \frac{2}{\lambda}e^{\phi}E_1 + 2e^{-2\phi}E_2, \\ A_- = \partial_{-\phi}H_1 + \lambda E_{-1} + E_{-2}. \end{cases} \quad (2.17)$$

Such A_{\pm} do just constitute a Lax pair representation of ZMS Eq.(2.1), which gives Eq.(2.1) as the zero-curvature equation $[\partial_+ - A_+, \partial_- - A_-] = 0$, the consistency condition of the equations for Ψ . In (2.17), H_1 , $E_{\pm 1}$ and $E_{\pm 2}$ are some 3×3 matrices defined as $H_1 = e_{11} - e_{33}$, $E_1 = e_{12} - e_{23}$, $E_2 = e_{31}$, $E_{-1} = e_{21} - e_{32}$ and $E_{-2} = e_{13}$ respectively. These matrices are among the independent generators of $SL(3, \mathcal{R})$ group [8].

Another aspect of the prolongation structure is the nonlinear realizations of the vector fields X_i and Y_i ($i = 0, 1$) in a finite-dimensional q -space. In Refs. [10,11], the authors cited some instances in illustration of the fact that the linear realizations of the vector fields are relevant to Lax representation while the nonlinear realizations of these fields associate themselves with the Bäcklund transformation of the considered nonlinear differential equation. We will show that this conclusion is also true for ZMS Eq.(2.1).

It is worthwhile to indicate that there does not exist an unpenetrable barrier between the linear realizations and the nonlinear realizations of the vector fields. In fact, there is a standard method to get the nonlinear realizations of the vector fields from their linear realizations, in which the Lax pair plays the crucial role [12]. Let us apply the method to ZMS model. Defining $q_1 = \psi_1/\psi_2$, $q_2 = -\psi_3/\psi_2$ as new pseudopotentials, we see from Lax pair (2.17) and auxiliary linear equations that these pseudopotentials [13] satisfy a set of Riccati-type equations:

$$\begin{aligned} \partial_+ q_1 &= \frac{2}{\lambda}(1 - q_1 q_2)e^{\phi}, & \partial_+ q_2 &= -2q_1 e^{-2\phi} - \frac{2}{\lambda}q_2^2 e^{\phi}, \\ \partial_- q_1 &= -(q_2 + \lambda q_1^2) + q_1 \pi_{\phi}, & \partial_- q_2 &= \lambda(1 - q_1 q_2) - q_2 \pi_{\phi}, \end{aligned} \quad (2.18)$$

where $\pi_{\phi} = \partial_{-\phi}$. On the other hand, it follows from (2.3) and (2.6) that

$$\begin{aligned} \partial_+ q_1 &= 2Y_0^1 e^{\phi} + 2Y_1^1 e^{-2\phi}, & \partial_+ q_2 &= 2Y_0^2 e^{\phi} + 2Y_1^2 e^{-2\phi}, \\ \partial_- q_1 &= X_0^1 + X_1^1 \pi_{\phi}, & \partial_- q_2 &= X_0^2 + X_1^2 \pi_{\phi}, \end{aligned} \quad (2.19)$$

in two-dimensional q -space. Therefore, the vector fields acquire the following nonlinear realizations:

$$\begin{cases} X_0 = -(q_2 + \lambda q_1^2)\partial_1 + \lambda(1 - q_1 q_2)\partial_2, \\ X_1 = q_1 \partial_1 - q_2 \partial_2, \\ Y_0 = \frac{1}{\lambda}(1 - q_1 q_2)\partial_1 - \frac{1}{\lambda}q_2^2 \partial_2, \\ Y_1 = -q_1 \partial_2. \end{cases} \quad (2.20)$$

Simultaneously, the prolonged algebra $a_2^{(2)}$ is nonlinearly realized as,

$$\begin{cases} H_1^{(m)} = \lambda^{-m}(q_1 \partial_1 - q_2 \partial_2), & E_1^{(m)} = -\lambda^{-m}[q_1^2 \partial_1 - (1 - q_1 q_2)\partial_2], \\ H_2^{(m)} = 3\lambda^{-m}(q_1 \partial_1 + q_2 \partial_2), & E_{-1}^{(m)} = \lambda^{-m}[(1 - q_1 q_2)\partial_1 - q_2^2 \partial_2], \\ E_2^{(m)} = -\lambda^{-m}q_2 \partial_1, & E_3^{(m)} = \lambda^{-m}[(1 + q_1 q_2)\partial_1 + q_2^2 \partial_2], \\ E_{-2}^{(m)} = -\lambda^{-m}q_1 \partial_2, & E_{-3}^{(m)} = -\lambda^{-m}[q_1^2 \partial_1 + (1 + q_1 q_2)\partial_2]. \end{cases} \quad (2.21)$$

In (2.20) and (2.21), λ is an arbitrary spectral parameter and m takes integer value. Notably, the vector fields (2.20) are not among a finite-dimensional subalgebra of $a_2^{(2)}$ unless λ equals to one. This is another important difference in the prolongation structure of ZMS equation from those of sine-Gordon equation, Ernst equation and chiral model [10].

III. THE BÄCKLUND TRANSFORMATION OF ZMS EQUATION

The aim of this section is to search the auto-Bäcklund transformation of ZMS Eq.(2.1) on the basis of the prolongation structure. We assume that the new ZMS field variables $\tilde{\phi}$ and $\tilde{\pi}_{\phi}$ are functions of the old ϕ , π_{ϕ} and the pseudopotentials q^a . The new forms α_i ($i = 1, 2$) which are got from the old ones by replacing ϕ and π_{ϕ} with $\tilde{\phi}(\phi, \pi_{\phi}, q^a)$ and $\tilde{\pi}_{\phi}(\phi, \pi_{\phi}, q^a)$ should vanish modulo the old α_i and the prolongation 1-forms Ω^a ; i.e., there should exist some 0-forms g_{ij} and 1-forms ν_i^a such that

$$\tilde{\alpha}_i(\tilde{\phi}, \tilde{\pi}_\phi) = g_{ij}\alpha_j(\phi, \pi_\phi) + \nu_i^a \wedge \Omega^a. \quad (3.1)$$

This is the condition for the existence of Bäcklund transformations. In terms of (2.2) and (2.3), the condition can be recast as

$$\partial_{\pi_\phi} \tilde{\phi} = 0, \quad \partial_\phi \tilde{\pi}_\phi = 0, \quad \tilde{\pi}_\phi = \pi_\phi(\partial_\phi + X_1)\tilde{\phi} + X_0\tilde{\phi}, \quad (3.2)$$

and

$$e^{\tilde{\phi}} - e^{-2\tilde{\phi}} = (e^\phi - e^{-2\phi})\partial_{\pi_\phi} \tilde{\pi}_\phi + (e^\phi Y_0 + e^{-2\phi} Y_1)\tilde{\pi}_\phi. \quad (3.3)$$

The special expressions of Eqs.(3.2) lead to the following ansatz solutions for the new ZMS field variables $\tilde{\phi}$ and $\tilde{\pi}_\phi$:

$$\begin{cases} \tilde{\phi} = c\phi + f(q^a), \\ \tilde{\pi}_\phi = \pi_\phi[c + X_1 f(q^a)] + X_0 f(q^a), \end{cases} \quad (3.4)$$

where c is an outstanding constant. Substituting (3.4) into (3.3) we get,

$$X_1 f(q^a) = c_1, \quad (3.5)$$

and,

$$e^{c\phi} e^{f(q^a)} - e^{-2c\phi} e^{-2f(q^a)} = e^\phi [c + c_1 + Y_0 X_0 f(q^a)] - e^{-2\phi} [c + c_1 - Y_1 X_0 f(q^a)]. \quad (3.6)$$

where c_1 is another constant.

We now apply the nonlinear expressions (2.20) of the vector fields X_i and Y_i ($i = 0, 1$) in the two-dimensional q -space to Eqs.(3.4)-(3.6). In this case, (3.5) becomes a first-order quasi-linear differential equation whose general solution reads:

$$f(q_1, q_2) = c_1 \ln q_1 + \omega(q_1 q_2), \quad (3.7)$$

where $\omega(q_1 q_2)$ is an arbitrary differentiable function of its variable $q_1 q_2$. Generally speaking, Eq.(3.5) would have another particular solution beyond (3.7). But we quit finding such a particular solution here. One of the reason is that there is no systematic method for searching it. What is more, even if we happened to find out a particular solution for Eq.(3.5), it would be excessive to expect this solution satisfying Eq.(3.6) further. If the Bäcklund transformation of ZMS equation exists, it is bound to connect with the general solution (3.7).

To determine the function $\omega(q_1 q_2)$ in (3.7) and then resolve completely the Bäcklund transformation for ZMS Eq.(2.1), Eq.(3.6) must be taken into account. After a simple and straightforward calculation we find,

$$c = 1, \quad c_1 = 0, \quad \omega(q_1 q_2) = \ln(2q_1 q_2 - 1). \quad (3.8)$$

Namely, the auto-Bäcklund transformation of ZMS Eq.(2.1) is as follows:

$$\tilde{\phi} = \phi + \ln(2q_1 q_2 - 1), \quad (3.9)$$

where the auxiliary pseudopotentials are determined by Riccati equations

$$\begin{aligned} \partial_+ q_1 &= \frac{2}{\lambda}(1 - q_1 q_2)e^\phi, & \partial_+ q_2 &= -2q_1 e^{-2\phi} - \frac{2}{\lambda}q_2^2 e^\phi, \\ \partial_- q_1 &= -(q_2 + \lambda q_1^2) + q_1 \partial_- \phi, & \partial_- q_2 &= \lambda(1 - q_1 q_2) - q_2 \partial_- \phi, \end{aligned} \quad (3.10)$$

for a given ϕ . One can easily justify that $\tilde{\phi}$ is a solution of ZMS Eq.(2.1) once ϕ fulfills Eq.(2.1) and vice versa. The problem of Bäcklund transformation of ZMS equation was ever discussed in [14,15]. The Bäcklund transformation obtained by Sharipov and Yamilov [14] is a set of second order differential equations, which is very cumbersome for solving single soliton solutions [15] and is difficult to turn to our Bäcklund transformation (3.9-3.10).

IV. DISCUSSIONS

In the previous sections we have studied the prolongation structure of Zhiber-Mikhailov-Shabat equation. The prolongation structure yields an incomplete set of commutators of vector fields in the pseudopotential space. Following Omote [10] and Ablowitz *et al* [12], we have found out the linear and nonlinear differential realizations of the vector fields respectively. It is shown that the linear realizations of the vector fields give the linear auxiliary equations (Lax pair) of ZMS equation, while their nonlinear realizations are connected with the Bäcklund transformation of the equation. Nevertheless, the application of this Bäcklund transformation to constructing new analytical solutions of ZMS equation from some old ones, *e.g.*, constructing the single-soliton solution from vacuum solution, remains an open problem. It is easy to see that ZMS Eq.(2.1) has an analytical solution governed by the first-order equations $\partial_{\pm}\phi = \mu^{\pm}\sqrt{2(2e^{\phi} + e^{-2\phi} - 3)}$ (μ is an arbitrary constant). But we have not driven out these equations from the Bäcklund transformation laws yet. Another even more interesting problem is perhaps to study the dressing group symmetry in ZMS model by virtue of prolongation structure. By some naive perception we conjecture that the finite dressing transformations of ZMS equation may be exposed through a slightly different nonlinear realization of the prolongation structure. The dressing procedure is a useful method for solving single soliton solution. The detail for such problems are now in preparation.

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- [1] R.K.Dodd, R.K.Bullough, Proc. Roy. Soc. Lond. **A352**, 481(1977).
- [2] A.V.Zhiber and A.B. Shabat, Dokl. Akad. Nauk USSR **247**, No.5(1979).
- [3] G.L.Jr Lamb, Rev. Mod. Phys. **43**, 99(1971).
- [4] A.V.Mikhailov, Pisma v ZhETF **30**, 443(1979).
- [5] A.G.Izergin and V.E.Korepin, Commun. Math. Phys. **79**, 303(1981).
- [6] F.A.Smirnov, Int. J. Mod. Phys. **A6**, 1407(1991).
- [7] C.J.Efthimiou, Nucl. Phys. **B398**, 697(1993).
- [8] Y.X.Chen, H.X.Yang and Z.M.Sheng, Phys. Lett. **B345**, 149(1995).
- [9] D.H.Wahlquist and F.B.Estabrook, J. Math. Phys. **16**, 1(1975).
- [10] M.Omote, J. Math. Phys. **27**, 2853(1986).
- [11] Y.Q.Lee and M.L.Ge, Commun. Theor. Phys. **10**, 79(1988).
- [12] M.J.Ablowitz, D.J.Kaup, A.C.Newell and H.Segur, Stud. Appl. Math. **53**, 249(1974).
- [13] M.C.Nucci, Riccati-type pseudopotentials and their applications in "Nonlinear Equations in the Applied Sciences", ed W.F.Ames and Colin Rogers, Academic Press (1992).
- [14] R.A.Sharipov and R.I.Yamilov, "Backlund transformation and the construction of an integrable boundary-value problem for the equation $u_{xt} = e^u - e^{-2u}$ " in: Problems of Mathematical Physics and Asymptotic Behavior of Their Solitons [in Russian] (Ufa 1991).
- [15] S.S.Safin and R.A.Sharipov, Theoretical Mathematical Physics, **94**, 462(1993).